

Anharmonic Oscillator Driven by Additive Ornstein–Uhlenbeck Noise

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We present an analytical study of a nonlinear oscillator subject to an additive Ornstein–Uhlenbeck noise. Known results are mainly perturbative and are restricted to the large dissipation limit (obtained by neglecting the inertial term) or to a quasi-white noise (i.e., a noise with vanishingly small correlation time). Here, in contrast, we study the small dissipation case (we retain the inertial term) and consider a noise with finite correlation time. Our analysis is non perturbative and based on a recursive adiabatic elimination scheme: a reduced effective Langevin dynamics for the slow action variable is obtained after averaging out the fast angular variable. In the conservative case, we show that the physical observables grow algebraically with time and calculate the associated anomalous scaling exponents and generalized diffusion constants. In the case of small dissipation, we derive an analytic expression of the stationary probability distribution function (PDF) which differs from the canonical Boltzmann–Gibbs distribution. Our results are in excellent agreement with numerical simulations.

KEY WORDS: Random processes; Fokker-Planck equations; colored noise; stochastic analysis methods.

1. INTRODUCTION

A theoretical description of the intrinsic thermal fluctuations induced by the coupling of a given dynamical system with a heat bath can be obtained by adding a noise term to the equations of motion. From this point of view that originates from Langevin's treatment of Brownian

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motion, the effect of the heat bath is modeled as a random force that transforms a deterministic motion into a stochastic dynamics in phase space. The physics of the system is then described by a probability distribution function (PDF) and the information about the observables is of statistical nature. When the additive noise is white (i.e., the correlation time of the noise is smaller than any intrinsic time scale of the system) and its amplitude satisfies the fluctuation-dissipation relation, the stationary PDF is identical to the canonical Boltzmann–Gibbs distribution.^(1,2)

A paradigm of a stochastic dynamical system for a quantitative study of the interplay between randomness and nonlinearity is provided by a particle trapped in a nonlinear confining potential and subject to a random noise.^(3,4) We showed in refs. 5–7 that in the limit of a vanishingly small damping rate, such a particle exhibits anomalous diffusion with exponents that depend on the form of the confining potential at infinity. For a non-zero damping rate, this anomalous diffusion occurs as intermediate time asymptotics: the particle diffuses in phase space and absorbs energy from the noise until the dissipative time scale is reached and the physical observables become stationary. For an additive Gaussian white noise, it was shown explicitly⁽⁶⁾ that, for times larger than the inverse damping rate, the intermediate time PDF matches the canonical distribution. For colored noise, we observed from numerical simulations that the anomalous growth exponents are different from those calculated for white noise. This behavior strongly suggests that the long time asymptotics in the case of colored noise is not identical to the canonical Boltzmann–Gibbs distribution. The physical reason for this change of behavior is the following: the intrinsic period of a nonlinear oscillator is a decreasing function of its energy and when the amplitude of the oscillations grows the period eventually becomes shorter than the correlation time of the noise. In that case, destructive interference between the fast variable and the noise suppresses the energy transfer from the noise to the system and the diffusion slows down. In this regime, the correlation time of the noise ceases to be the shortest time scale in the system and the noise can not be treated perturbatively as white. Therefore, usual perturbative calculations based on small correlation time expansions^(8–10) cannot predict the correct colored noise scalings (this was shown explicitly for the pendulum with multiplicative noise in ref. 11). Besides, the averaging procedure used for the white noise fails for colored noise and in ref. 6 we could only infer the colored noise exponents using qualitative scaling arguments.

Our aim, in this work, is to perform an analytical study of nonlinear oscillator subject to an additive Ornstein–Uhlenbeck noise and to derive quantitative results for its long time behavior. We need to generalize the averaging method used for white noise in ref. 5, 6 so that it can be applied

to a regime where the correlation time of the noise is not the shortest time scale in the system. We define, in a recursive manner, new coordinates on phase space that embody, order by order, correlations between the noise and the fast angular variable and then average out the fast variable. This technique allows us to obtain an analytical expression for the long time asymptotic behavior of the PDF of the system, to derive the anomalous scaling exponents and to calculate generalized diffusion constants. In particular, it is shown explicitly that the long time PDF of a nonlinear oscillator subject to additive colored noise is not the canonical Boltzmann–Gibbs distribution. Our results are systematically compared with numerical simulations.

The outline of this paper is as follows. In Section 2, we review results derived in ref. 6 and which are required for the present work. In Section 3 we first explain why the straightforward averaging technique fails for colored noise, provide a heuristic calculation that explains the idea underlying the correct averaging scheme and develop it analytically. An effective averaged Langevin system for the slow variables is derived in the Section 3.4. In Section 4, we derive analytical expressions for the PDF, the anomalous diffusion exponents and the generalized diffusion constants; we also extend our calculations to a weakly dissipative system and compare our results with numerical simulations. Section 5 is devoted to concluding remarks. Technical mathematical details are given in the appendices. In particular, in Appendix D, we discuss the ‘Best Fokker–Planck Equation’ (BFPE) approximation,^(12,13) based on partial resummation to all orders in the correlation time τ of a perturbative expansion for small noise amplitude.

2. REVIEW OF EARLIER RESULTS

Consider a nonlinear oscillator of amplitude $x(t)$, trapped in a confining potential $\mathcal{U}(x)$ and subject to an additive noise $\xi(t)$:

$$\frac{d^2}{dt^2}x(t) = -\frac{\partial\mathcal{U}(x)}{\partial x} + \xi(t). \quad (1)$$

The statistical properties of the random function $\xi(t)$ need not be specified at this stage. We restrict our analysis to the case where the potential \mathcal{U} is an even polynomial in x for $|x| \rightarrow \infty$. A suitable rescaling of x allows us to write

$$\mathcal{U} \sim \frac{x^{2n}}{2n} \quad \text{with } n \geq 2. \quad (2)$$

As the amplitude $x(t)$ of the oscillator grows with time, only the asymptotic behavior of $\mathcal{U}(x)$ for $|x| \rightarrow \infty$ is relevant and Eq. (1) reduces to

$$\frac{d^2}{dt^2}x(t) + x(t)^{2n-1} = \xi(t). \quad (3)$$

We emphasize that all stochastic differential equations involving white noise are interpreted in the Stratonovich sense, that allows the use of ordinary differential calculus for the change of variables. We review here some previously derived results that are required for the present work.

2.1. Energy-Angle Coordinates

A key feature of Eq. (3) is that its underlying deterministic system (obtained by setting $\xi \equiv 0$), is *integrable*. The associated energy and the angle variable are given by^(5,6)

$$E = \frac{1}{2}\dot{x}^2 + \frac{1}{2n}x^{2n}, \quad \text{and} \quad \phi = \frac{\sqrt{n}}{(2n)^{1/2n}} \int_0^{x/E^{1/2n}} \frac{du}{\sqrt{1 - \frac{u^{2n}}{2n}}}, \quad (4)$$

where the angle ϕ is defined modulo the oscillation period $4K_n$, with

$$K_n = \sqrt{n} \int_0^1 \frac{du}{\sqrt{1 - u^{2n}}}. \quad (5)$$

In terms of the energy-angle coordinates (E, ϕ) , the original variables (x, \dot{x}) read as follows:

$$x = E^{1/2n} \mathcal{S}_n(\phi), \quad (6)$$

$$\dot{x} = (2n)^{\frac{n-1}{2n}} E^{1/2} \mathcal{S}'_n(\phi), \quad (7)$$

where the hyperelliptic function \mathcal{S}_n is defined as^(14,15)

$$\mathcal{S}_n(\phi) = Y \Leftrightarrow \phi = \frac{\sqrt{n}}{(2n)^{1/2n}} \int_0^Y \frac{du}{\sqrt{1 - \frac{u^{2n}}{2n}}}. \quad (8)$$

The presence of external noise spoils the integrability of the dynamical system (3) but does not preclude the use of (E, ϕ) instead of (x, \dot{x}) as coordinates in phase space. Introducing the auxiliary variable Ω defined as

$$\Omega = (2n)^{\frac{n+1}{2n}} E^{\frac{1}{2}}, \quad (9)$$

Eq. (3) is written as a system of two coupled stochastic differential equations⁽⁶⁾

$$\dot{\Omega} = n \mathcal{S}'_n(\phi) \xi(t), \quad (10)$$

$$\dot{\phi} = \left(\frac{\Omega}{(2n)^{\frac{1}{2n}}} \right)^{\frac{n-1}{n}} - \frac{\mathcal{S}_n(\phi)}{\Omega} \xi(t). \quad (11)$$

This system is rigorously equivalent to the original problem (Eq. 3) and has been derived without any hypothesis on the nature of the driving force $\xi(t)$ which may even be a deterministic function or may assume arbitrary statistical properties. This external perturbation $\xi(t)$ continuously injects energy into the system. We have shown in ref. 6 that, when ξ is a Gaussian white noise or a dichotomous Poisson noise, the typical value of Ω grows algebraically with time. We have also verified numerically that the same behavior is true for an Ornstein–Uhlenbeck noise. As seen from Eq. (11), the growth of the phase ϕ is faster than that of Ω . Assuming that in the long time limit ϕ is uniformly distributed over a period $[0, 4K_n]$, we obtain, on averaging Eqs. (6) and (7) over the angle variable, the following equipartition relations

$$\langle E \rangle = \frac{n+1}{2n} \langle \dot{x}^2 \rangle, \quad (12)$$

$$\langle \dot{x}^2 \rangle = \langle x^{2n} \rangle. \quad (13)$$

These relations are in excellent agreement with the numerical simulations as shown in Fig. 1.

2.2. The White Noise Case

In this subsection, we consider $\xi(t)$ to be a Gaussian white noise of zero mean value and of amplitude \mathcal{D} :

$$\begin{aligned} \langle \xi(t) \rangle &= 0, \\ \langle \xi(t) \xi(t') \rangle &= \mathcal{D} \delta(t - t'). \end{aligned} \quad (14)$$

As described in ref. 6, an averaging procedure over the angle variable ϕ allows us to derive a closed effective Langevin equation for the slow variable Ω . Starting with the full Fokker–Planck equation for the evolution of

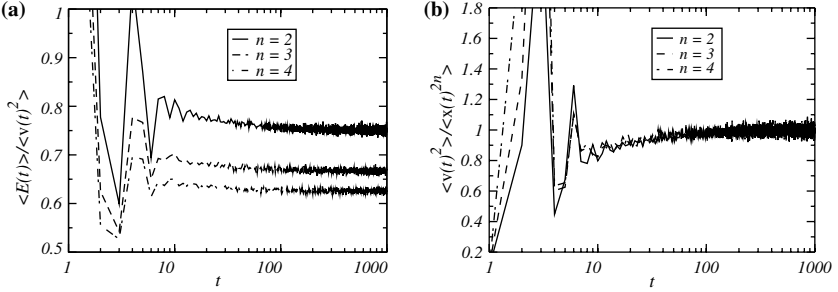


Fig. 1. Equipartition relations with Ornstein-Uhlenbeck noise. Equations (3) and (20) are integrated numerically for $\mathcal{D}=1$, $\tau=5$, with a timestep $\delta t=10^{-5}$ and averaged over 10^4 realizations. (a) The first equipartition ratio $\langle E(t) \rangle / \langle v(t)^2 \rangle$ is close to the theoretical value $\frac{n+1}{2n}$ given in Eq. (12): $3/4$ for $n=2$; $2/3$ for $n=3$; $5/8$ for $n=4$. (b) The second equipartition ratio $\langle v(t)^2 \rangle / \langle x(t)^{2n} \rangle$ is close to 1 for $n=2, 3, 4$.

the PDF $P_t(\Omega, \phi)$ associated with the system (10) and (11) and integrating out the angular variable ϕ , we obtain a phase-averaged Fokker-Planck equation for the marginal distribution $\tilde{P}_t(\Omega)$:

$$\partial_t \tilde{P} = \frac{\tilde{\mathcal{D}}}{2} \left(\partial_\Omega^2 \tilde{P} - \frac{1}{n} \partial_\Omega \frac{\tilde{P}}{\Omega} \right), \quad \text{with} \quad \tilde{\mathcal{D}} = \frac{n^2 (2n)^{\frac{1}{n}}}{n+1} \mathcal{D}. \quad (15)$$

The effective Langevin dynamics for the variable Ω is thus

$$\dot{\Omega} = \frac{\tilde{\mathcal{D}}}{2n} \frac{1}{\Omega} + \tilde{\xi}(t), \quad (16)$$

where $\tilde{\mathcal{D}}$ is the amplitude of the effective Gaussian white noise $\tilde{\xi}(t)$. It is readily deduced from Eq. (16) that the variable Ω has a normal diffusive behavior with time:

$$\langle \Omega^2 \rangle = \frac{(2n)^{\frac{n+1}{n}} \mathcal{D}}{2} t. \quad (17)$$

The averaged Fokker-Planck equation (15) is exactly solvable, leading to the following expression for the energy PDF

$$\tilde{P}_t(E) = \frac{1}{\Gamma\left(\frac{n+1}{2n}\right)} \frac{1}{E} \left(\frac{(2n)^{\frac{n+1}{n}} E}{2\tilde{\mathcal{D}}t} \right)^{\frac{n+1}{2n}} \exp \left\{ -\frac{(2n)^{\frac{n+1}{n}} E}{2\tilde{\mathcal{D}}t} \right\}, \quad (18)$$

$\Gamma(\cdot)$ being the Euler Gamma function.⁽¹⁴⁾ The asymptotic time dependence of all moments of the energy, amplitude or velocity is calculated analytically from Eqs. (9), (12), (13), and (18), leading to the long time scaling behavior of dynamical observables as well as analytical expressions for the prefactors (generalized diffusion constants)

$$\begin{aligned}\langle E \rangle &= \frac{\mathcal{D}}{2}t, \\ \langle \dot{x}^2 \rangle &= \frac{n\mathcal{D}}{n+1}t, \\ \langle x^2 \rangle &= \frac{\Gamma\left(\frac{3}{2n}\right)}{\Gamma\left(\frac{1}{2n}\right)} \left(\frac{2n^2}{n+1}\mathcal{D}t\right)^{\frac{1}{n}}.\end{aligned}\quad (19)$$

The physical observables grow algebraically with time and the corresponding anomalous diffusion exponents depend only on the confining potential at infinity. These quantitative results are in excellent agreement with numerical simulations.⁽⁶⁾

2.3. Scaling Behavior for Colored Noise

We now take $\xi(t)$ to be a colored Gaussian noise with non-zero correlation time τ and generated from the Ornstein-Uhlenbeck equation

$$\frac{d\xi(t)}{dt} = -\frac{1}{\tau}\xi(t) - \frac{1}{\tau}\eta(t), \quad (20)$$

where $\eta(t)$ is a Gaussian white noise of zero mean value and of amplitude \mathcal{D} . In the stationary limit, when $t, t' \gg \tau$, we have:

$$\langle \xi(t) \rangle = 0 \text{ and } \langle \xi(t)\xi(t') \rangle = \frac{\mathcal{D}}{2\tau} e^{-|t-t'|/\tau}. \quad (21)$$

In ref. 6, we derived from a self-consistent scaling Ansatz that, when the noise has a finite correlation time τ , the variable Ω has a sub-diffusive behavior and grows as

$$\Omega \sim \left(\frac{\mathcal{D}t}{2\tau^2}\right)^{\frac{n}{2(2n-1)}}. \quad (22)$$

From this equation and Eqs. (9), (12) and (13), the following scaling relations are deduced

$$\begin{aligned} E &\sim \left(\frac{\mathcal{D}t}{2\tau^2}\right)^{\frac{n}{(2n-1)}}, \\ \dot{x} &\sim \left(\frac{\mathcal{D}t}{2\tau^2}\right)^{\frac{n}{2(2n-1)}}, \\ x &\sim \left(\frac{\mathcal{D}t}{2\tau^2}\right)^{\frac{1}{2(2n-1)}}. \end{aligned} \quad (23)$$

We observe that the exponents associated with colored noise are different from those obtained for white noise. We emphasize that Eq. (23) gives only qualitative scaling relations that were obtained only by heuristic arguments. In particular, the prefactors (which are dimensionless pure numbers) are not known at this stage. In the following sections, we solve this problem analytically: we calculate the PDF of the anharmonic oscillator subject to Ornstein–Uhlenbeck noise and derive explicit and complete formulae for the moments of the dynamical observables in the long time limit.

The result of Eq. (23) is supported by numerical simulations: the colored noise scalings are observed for large enough times, even for an arbitrarily small correlation time τ (see Fig. 2). The crossover between the white noise scalings (19) and the colored noise scalings (23) occurs when the period T of the underlying deterministic oscillator (which is a decreasing function of its amplitude) is of the order of τ , i.e., for a typical time $t_c \sim \mathcal{D}^{-1} \tau^{-2n/(n-1)}$. For $t \ll t_c$, the angular period of the system is large as compared to the correlation time of the noise which, therefore, acts as if it were white; for $t \gg t_c$, the noise is highly correlated over a period and is smeared out, leading to a slower diffusion.

Some interesting qualitative results have thus already been obtained. But, a quantitative study of the nonlinear oscillator (3) with Ornstein–Uhlenbeck noise has remained out of reach. The reason is that the averaging technique, that applies perfectly for the white noise, leads to erroneous results when applied to the colored noise. We now explain why the naive averaging method fails and how to modify it to make it work.

3. AVERAGING METHOD FOR ORNSTEIN–UHLENBECK NOISE

A closed Fokker–Planck equation for the probability distribution function $P_t(\Omega, \phi)$ on the system’s phase space can not be derived when

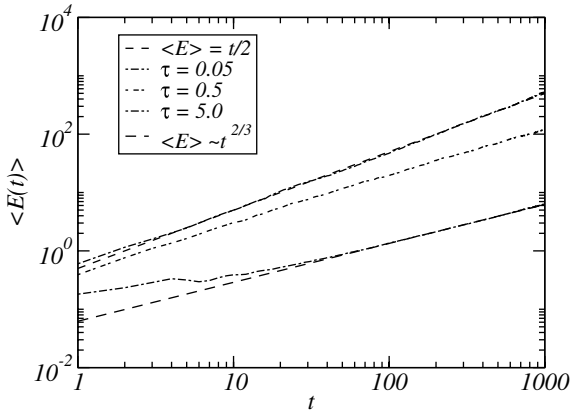


Fig. 2. Asymptotic behavior of the nonlinear oscillator with additive Ornstein–Uhlenbeck noise. Eqs. (3) and (20) are integrated numerically for $\mathcal{D}=1$, $n=2$, with a timestep $\delta t=10^{-5}$ and averaged over 10^3 realizations. For $\tau=0.05$, 0.5 and 5.0 , we plot the average $\langle E \rangle$ versus time t . The dashed lines correspond to the analytical predictions, $\langle E \rangle = \mathcal{D}t/2$ for white noise (see Eq. (19)) and $\langle E \rangle = 0.533(\mathcal{D}t/\tau^2)^{2/3}$ for colored noise with $\tau=5.0$ (see Eq. (61)).

the noise is colored: because the noise is correlated in time, the system’s dynamics acquires a non-Markovian character and the evolution equation for $P_t(\Omega, \phi)$, which can be derived using functional methods, ref. (8, 9, 16–20), involves correlation functions at different times. This hierarchy can not be closed at any finite order unless truncation approximations are used.^(18,21) However, by embedding the initial problem into a higher dimensional Markovian system, a rigorous description in terms of a Fokker–Planck evolution equation can be formulated. Indeed, the random oscillator described by Eqs. (10) and (11) together with the Ornstein–Uhlenbeck noise generated by Eq. (20) forms a three dimensional stochastic system driven by a white noise $\eta(t)$. The stochastic Eq. (10), (11) and (20) define a random dynamical system in the three dimensional space (Ω, ϕ, ξ) , the three variables being treated on an equal footing:

$$\dot{\Omega} = n S'_n(\phi) \xi, \quad (24)$$

$$\dot{\phi} = \left(\frac{\Omega}{(2n)^{\frac{1}{2n}}} \right)^{\frac{n-1}{n}} - \frac{S_n(\phi)}{\Omega} \xi, \quad (25)$$

$$\dot{\xi} = -\frac{1}{\tau} \xi - \frac{1}{\tau} \eta(t), \quad (26)$$

where $\eta(t)$ is a Gaussian white noise of amplitude \mathcal{D} . The Fokker–Planck equation for the joint PDF $P_t(\Omega, \phi, \xi)$ is then given by

$$\begin{aligned} \frac{\partial P_t}{\partial t} = & -n \frac{\partial}{\partial \Omega} (\mathcal{S}'_n(\phi) \xi P_t) - \frac{\partial}{\partial \phi} \left(\left(\left(\frac{\Omega}{(2n)^{\frac{1}{2n}}} \right)^{\frac{n-1}{n}} - \frac{\mathcal{S}_n(\phi) \xi}{\Omega} \right) P_t \right) \\ & + \frac{1}{\tau} \frac{\partial \xi P_t}{\partial \xi} + \frac{\mathcal{D}}{2\tau^2} \frac{\partial^2 P_t}{\partial \xi^2}. \end{aligned} \quad (27)$$

If an exact solution of Eq. (27) could be found, the PDF on the original phase space (Ω, ϕ) will then be obtained by integrating the full solution over the variable ξ . Unfortunately, this does not seem to be feasible. However, the evolution of the angular variable ϕ is fast as compared to that of Ω irrespective of the correlation time of the noise. Therefore, it is desirable to average out ϕ from the dynamics in order to obtain an effective Langevin equation for the slow variable Ω , as has been done in the white noise case.

3.1. Failure of Straightforward Averaging

We now average the Fokker–Planck equation (27) over the angular variable ϕ assuming that for $t \rightarrow \infty$ the probability measure for ϕ is uniform over the interval $[0, 4K_n]$. Using the fact that the average of the derivative of any function with respect to ϕ is zero:

$$\overline{\partial_\phi(\dots)} = 0, \quad (28)$$

the phase-average of the Fokker–Planck equation (27) governing the evolution of the marginal distribution $\tilde{P}_t(\Omega, \xi)$ is found to be

$$\frac{\partial \tilde{P}_t}{\partial t} = \frac{1}{\tau} \frac{\partial \xi \tilde{P}_t}{\partial \xi} + \frac{\mathcal{D}}{2\tau^2} \frac{\partial^2 \tilde{P}_t}{\partial \xi^2}; \quad (29)$$

this equation corresponds to the following Langevin dynamics for the couple of variables (Ω, ξ)

$$\begin{aligned} \dot{\Omega} &= 0 \\ \dot{\xi} &= -\frac{1}{\tau} \xi - \frac{1}{\tau} \eta(t). \end{aligned} \quad (30)$$

This result implies that Ω has a trivial dynamics: it is no more stochastic and is conserved. The integration over the angular variable averages out

the noise itself and leads to conclusions that are blatantly wrong. This failure is due to the destructive interference between the fast angular variable and the noise: a simplistic adiabatic elimination of ϕ eliminates the noise too. The failure of this naive averaging procedure can also be understood from a direct analysis of Eqs. (24), (25) and (26): the functions $\mathcal{S}_n(\phi)$ and $\mathcal{S}'_n(\phi)$ are periodic in ϕ with period $4K_n$ and have a zero mean value. Thus, if we average Eqs. (24) and (26) over ϕ and discard the correlations between ϕ and ξ , the erroneous dynamics (30) is obtained. A correct averaging scheme must take into account the fact that the colored noise ξ is not constant but varies during the period T of the underlying deterministic oscillator (the angle ϕ covers the interval $[0, 4K_n]$ during T); when $T \ll \tau$, the variation of ξ can be estimated from its time derivative that contains a white noise term (see Eq. (20)) which will survive the averaging procedure.

3.2. A Heuristic Calculation

The preceding discussion suggests that we must transform the system (24), (25) and (26) so as to make the time derivative of ξ appear explicitly in the evolution Eq. (24) of the slow variable Ω . Before performing this transformation in a systematic manner, we explain the heuristics of the method. Taking into account the fact that Ω grows to infinity with time, let us simplify Eq. (25) to

$$\dot{\phi} \simeq \left(\frac{\Omega}{(2n)^{\frac{1}{2n}}} \right)^{\frac{n-1}{n}}, \quad (31)$$

and use this relation to transform Eq. (24) as follows

$$\dot{\Omega} = \frac{n\mathcal{S}'_n(\phi)\dot{\phi}\xi}{\dot{\phi}} \simeq \frac{n\mathcal{S}'_n(\phi)\dot{\phi}\xi}{\left(\frac{\Omega}{(2n)^{\frac{1}{2n}}} \right)^{\frac{n-1}{n}}}. \quad (32)$$

Introducing a new variable Y defined as

$$Y = \frac{\Omega^{\frac{2n-1}{n}}}{(2n)^{\frac{n-1}{2n^2}}}, \quad (33)$$

we rewrite Eq. (32) as

$$\frac{1}{2n-1} \dot{Y} = \mathcal{S}'_n(\phi)\dot{\phi}\xi = \frac{d(\mathcal{S}_n(\phi)\xi)}{dt} - \mathcal{S}_n(\phi)\dot{\xi}. \quad (34)$$

Using Ornstein–Uhlenbeck’s equation (26), Eq. (34) becomes

$$\frac{d}{dt} \left(\frac{Y}{2n-1} - \mathcal{S}_n(\phi)\xi \right) = \frac{\mathcal{S}_n(\phi)\xi}{\tau} + \mathcal{S}_n(\phi)\frac{\eta}{\tau}. \quad (35)$$

In the long time limit, the term $\mathcal{S}_n(\phi)\xi$ which remains of finite order becomes negligible as compared to Y that grows as a power-law with time. Besides, on the right hand side of Eq. (35) we also neglect the colored noise ξ with respect to the white noise η . We thus conclude that Y has a normal diffusive behavior and scales as

$$\langle Y^2 \rangle \propto \frac{\mathcal{D}t}{2\tau^2}. \quad (36)$$

This scaling result is equivalent to Eq. (22) and implies Eq. (23). The variable Y undergoes an effective diffusive dynamics and plays a role similar to that of Ω in the white noise case. In order to obtain a quantitative agreement between theory and numerics, we must set this heuristic calculation on a sound basis. In particular, some terms that were overlooked in the sketchy discussion above contribute to the long time behavior of the PDF in phase space and must be taken into account.

3.3. Coordinate Transformations

A correct averaging procedure is performed by defining recursively new sets of dynamical variables on the global three dimensional phase space (Ω, ϕ, ξ) . These new variables will embody the correlations between ϕ and ξ and provide a book-keeping device that transforms the heuristic calculation described above into a systematic perturbative expansion. We show in the Appendix B that a suitable set of variables is given by (Z, ϕ, ξ) , where Z is defined as

$$Z(Y, \phi, \xi) = Y - (2n-1) \left(\mathcal{S}_n(\phi)\xi + \frac{(2n)^{\frac{n-1}{2n(2n-1)}} C_n(\phi)\xi}{\tau Y^{\frac{n-1}{2n-1}}} + \frac{\mathcal{S}_n^2(\phi)\xi^2}{2Y} \right); \quad (37)$$

the variable Y was defined in Eq. (33) and the function $C_n(\phi)$ satisfies

$$C'_n(\phi) = \frac{dC_n(\phi)}{d\phi} = \mathcal{S}_n(\phi) \quad \text{and} \quad \overline{C_n(\phi)} = \int_0^{4K_n} C_n(\phi) d\phi = 0, \quad (38)$$

the overline indicates an average over the angular period.

Equation (37) defines a change of dynamical variables between the set (Ω, ϕ, ξ) and the set (Z, ϕ, ξ) . The system (24), (25) and (26) must now be expressed in terms of the new variables and we obtain

$$\frac{1}{2n-1} \dot{Z} = J_Z(Z, \phi, \xi) + \mathcal{D}_Z(Z, \phi, \xi) \frac{\eta(t)}{\tau}, \quad (39)$$

$$\dot{\phi} = J_\phi(Z, \phi, \xi), \quad (40)$$

$$\dot{\xi} = -\frac{1}{\tau} \xi - \frac{1}{\tau} \eta(t). \quad (41)$$

The current and diffusion functions that appear in Eqs. (39) and (40) are given by

$$J_Z(Z, \phi, \xi) = \frac{(2n)^{\frac{n-1}{2n(2n-1)}} C_n(\phi) \xi}{\tau Z^{\frac{n-1}{2n-1}}} + \frac{2\mathcal{S}_n^2(\phi) + (n-1)\mathcal{S}'_n(\phi)C_n(\phi)}{\tau Z} \xi^2, \quad (42)$$

$$\mathcal{D}_Z(Z, \phi, \xi) = \mathcal{S}_n(\phi) + \frac{(2n)^{\frac{n-1}{2n(2n-1)}} C_n(\phi)}{\tau Z^{\frac{n-1}{2n-1}}} + \frac{\mathcal{S}_n^2(\phi) \xi}{Z}, \quad (43)$$

$$J_\phi(Z, \phi, \xi) = \left(\frac{\Omega(Z, \phi, \xi)}{(2n)^{\frac{1}{2n}}} \right)^{\frac{n-1}{n}} - \frac{\mathcal{S}_n(\phi)}{\Omega(Z, \phi, \xi)} \xi. \quad (44)$$

In Appendix B, we show that the stochastic dynamical system (39), (40) and (41) is obtained from the original Eqs. (24), (25) and (26) by retaining all contributions up to the order $\mathcal{O}(Z^{-1})$. The crucial difference between the two systems appears when Eq. (24) is compared with Eq. (39): the evolution of Z contains an explicit white noise contribution that embodies the variations of the colored noise ξ during a time scale shorter than the correlation time τ . This white noise term will survive the averaging process and will allow us to derive a nontrivial Langevin dynamics for Z .

3.4. Averaged Langevin System

As for the white noise problem, we shall eliminate the fast angular variable in the Fokker–Planck equation for the PDF $\Pi_t(Z, \phi, \xi)$ associated with Eqs. (39, 40 and 41). This Fokker–Planck equation is given by

$$\begin{aligned}
\frac{\partial \Pi_t}{\partial t} &= -(2n-1) \frac{\partial}{\partial Z} (J_Z \Pi_t) - \frac{\partial}{\partial \phi} (J_\phi \Pi_t) + \frac{1}{\tau} \frac{\partial \xi \Pi_t}{\partial \xi} \\
&+ \frac{\mathcal{D}}{2\tau^2} \left\{ (2n-1)^2 \frac{\partial}{\partial Z} \mathcal{D}_Z \frac{\partial}{\partial Z} (\mathcal{D}_Z \Pi_t) - (2n-1) \frac{\partial^2}{\partial \xi \partial Z} (\mathcal{D}_Z \Pi_t) \right. \\
&\left. - (2n-1) \frac{\partial}{\partial Z} \mathcal{D}_Z \frac{\partial \Pi_t}{\partial \xi} + \frac{\partial^2 \Pi_t}{\partial \xi^2} \right\}. \tag{45}
\end{aligned}$$

Integrating out the fast variable ϕ , we obtain the following averaged Fokker–Planck equation that describes the evolution of the marginal PDF $\tilde{\Pi}_t(Z, \xi)$ (the details of the calculations are explained in Appendix C)

$$\frac{\partial \tilde{\Pi}_t}{\partial t} = -(2n-1)\mu \frac{\partial}{\partial Z} \left(\frac{(3-n)\xi^2}{\tau Z} \tilde{\Pi}_t \right) + \frac{1}{\tau} \frac{\partial \xi \tilde{\Pi}_t}{\partial \xi} \tag{46}$$

$$\begin{aligned}
&+ \frac{\mathcal{D}}{2\tau^2} \left\{ (2n-1)^2 \mu \frac{\partial^2 \tilde{\Pi}_t}{\partial Z^2} - (2n-1)\mu \frac{\partial^2}{\partial \xi \partial Z} \left(\frac{\xi \tilde{\Pi}_t}{Z} \right) \right. \\
&\left. - (2n-1)\mu \frac{\partial}{\partial Z} \left(\frac{\xi}{Z} \frac{\partial \tilde{\Pi}_t}{\partial \xi} \right) + \frac{\partial^2 \tilde{\Pi}_t}{\partial \xi^2} \right\}, \tag{47}
\end{aligned}$$

where the parameter μ , calculated in Appendix A, is given by

$$\mu = \overline{S_n^2} = (2n)^{\frac{1}{n}} \frac{\Gamma\left(\frac{3}{2n}\right) \Gamma\left(\frac{n+1}{2n}\right)}{\Gamma\left(\frac{1}{2n}\right) \Gamma\left(\frac{n+3}{2n}\right)}. \tag{48}$$

Let us now consider the following Langevin system for the variables Z and ξ :

$$\dot{Z} = (2n-1)\mu \frac{(3-n)\xi^2}{\tau Z} + (2n-1) \frac{\sqrt{\mu}}{\tau} \tilde{\eta}(t) + (2n-1)\mu \frac{\xi}{\tau Z} \eta(t), \tag{49}$$

$$\dot{\xi} = -\frac{1}{\tau} \xi - \frac{1}{\tau} \eta(t), \tag{50}$$

where $\tilde{\eta}(t)$ and $\eta(t)$ are two independent Gaussian white noises of amplitude \mathcal{D} . Writing the Fokker–Planck equation related to this stochastic system (49) and (50) we retrieve the averaged Fokker–Planck equation (47) up to terms of the order $\mathcal{O}(Z^{-2})$ (recall that the stochastic Eqs. (39), (40) and (41) are equivalent to Eqs. (24), (25) and (26) at order $\mathcal{O}(Z^{-1})$). We have thus found the effective stochastic Langevin equation (49) for the slow variable Z coupled with the Ornstein–Uhlenbeck noise ξ . This dynamics is

however not exactly solvable and we need one last transformation before deriving explicit analytical results. In terms of the new variable

$$Z_1 = Z + (2n - 1)\mu \frac{\xi^2}{2Z}, \quad (51)$$

Eq. (49) becomes

$$\dot{Z}_1 = \dot{Z} + (2n - 1)\mu \frac{\xi \dot{\xi}}{2Z} = (2n - 1)\mu \frac{(2 - n)\xi^2}{\tau Z_1} + (2n - 1)\frac{\sqrt{\mu}}{\tau} \tilde{\eta}(t), \quad (52)$$

where we have neglected all terms of order strictly less than $\mathcal{O}(Z^{-1})$. Recalling that ξ^2 has a finite mean value equal to $\mathcal{D}/2\tau$, we rewrite the Langevin equation (52) for Z_1 as follows

$$\dot{Z}_1 = (2n - 1)(2 - n)\frac{\mu\mathcal{D}}{2\tau^2} \frac{1}{Z_1} + (2n - 1)\frac{\sqrt{\mu}}{\tau} \tilde{\eta}(t) + (2n - 1)(2 - n)\frac{\mu}{\tau Z_1} (\xi^2 - \langle \xi^2 \rangle). \quad (53)$$

This equation contains two independent noise sources: a white noise contribution, $\tilde{\eta}(t)$, and a non-Gaussian colored noise term, $(\xi^2 - \langle \xi^2 \rangle)$, of zero mean and of finite variance $\mathcal{D}^2/2\tau^2$. This colored noise is multiplied by a factor proportional to $1/Z_1$ and therefore its effect becomes negligible as compared to that of $\tilde{\eta}(t)$ when $t \rightarrow \infty$. We thus simplify Eq. (53) to

$$\dot{Z}_1 = (2n - 1)(2 - n)\frac{\mu\mathcal{D}}{2\tau^2} \frac{1}{Z_1} + (2n - 1)\frac{\sqrt{\mu}}{\tau} \tilde{\eta}(t). \quad (54)$$

We have thus obtained an effective white noise Langevin dynamics for the variable Z_1 . This equation plays the same role for the nonlinear oscillator subject to a colored Ornstein–Uhlenbeck noise as that played by Eq. (16) for white noise. The mathematical structure of the two Eqs. (16) and (54) is the same and they differ only by the constant prefactors that embody the specific parameters of each problem.

4. ANALYTICAL RESULTS IN THE LONG TIME LIMIT

In this section, we solve the effective dynamics for the variable Z_1 and deduce statistical results for the long time behavior of the energy, the amplitude and the velocity of the nonlinear oscillator driven by an

Ornstein–Uhlenbeck process. We shall not only retrieve the scalings predicted in Eq. (23) but also derive explicit formulae for the prefactors and for the skewness and the flatness of the energy PDF. All these analytical results compare favorably with numerical simulations. Finally, we show that our analysis can be extended to include a weak dissipation term in the dynamics.

4.1. Calculation of the PDF

The Fokker–Planck equation associated with the effective Langevin dynamics (54) of Z_1 is exactly solvable (using the method of ref. 5) and we obtain the following expression for the asymptotic PDF

$$P_t(Z_1) = \frac{2}{\Gamma\left(\frac{n+1}{2(2n-1)}\right)} \left(\frac{\tau^2}{2(2n-1)^2 \mu D t} \right)^{\frac{n+1}{2(2n-1)}} Z_1^{\frac{2-n}{2n-1}} \exp\left(-\frac{\tau^2 Z_1^2}{2(2n-1)^2 \mu D t}\right). \quad (55)$$

Expressing the energy E as a function of Z_1 with the help of Eqs. (9), (33), (37) and (51), we obtain at the leading order

$$E = \left(\frac{Z_1}{2n}\right)^{\frac{2n}{2n-1}} \quad \text{and thus} \quad dE = \frac{1}{2n-1} \left(\frac{Z_1}{2n}\right)^{\frac{1}{2n-1}} dZ_1. \quad (56)$$

From Eq. (55), we deduce the asymptotic expression of the PDF of the energy

$$P_t(E) = \frac{\frac{2n-1}{n}}{\Gamma\left(\frac{n+1}{4n-2}\right)} \left(\frac{2n^2 \tau^2}{(2n-1)^2 \mu D t} \right)^{\frac{n+1}{4n-2}} E^{\frac{1-2n}{2n}} \exp\left(-\frac{2n^2 \tau^2 E^{\frac{2n-1}{n}}}{(2n-1)^2 \mu D t}\right). \quad (57)$$

The statistical behavior of the position, velocity and energy of the system in the long time limit can now be derived by using the PDF (57) and Eqs. (6) and (7):

$$\langle E \rangle = \frac{\Gamma\left(\frac{3n+1}{4n-2}\right)}{\Gamma\left(\frac{n+1}{4n-2}\right)} \left(\frac{(2n-1)^2 \mu D t}{2n^2 \tau^2} \right)^{\frac{n}{2n-1}}, \quad (58)$$

$$\langle \dot{x}^2 \rangle = \frac{2n}{n+1} \langle E \rangle, \quad (59)$$

$$\langle x^2 \rangle = \mu \langle E^{\frac{1}{n}} \rangle = \frac{\Gamma\left(\frac{n+3}{4n-2}\right)}{\Gamma\left(\frac{n+1}{4n-2}\right)} \left(\frac{(2n-1)^2 \mu^{2n} \mathcal{D}t}{2n^2 \tau^2} \right)^{\frac{1}{2n-1}}. \quad (60)$$

In the last equality we used $\mu = \overline{\mathcal{S}_n^2}$, according to Eq. (48). In particular, we find

For $n=2$,

$$\langle E \rangle = 0.533 \left(\frac{\mathcal{D}t}{\tau^2} \right)^{2/3}, \quad \langle \dot{x}^2 \rangle = 0.711 \left(\frac{\mathcal{D}t}{\tau^2} \right)^{2/3}, \quad \langle x^2 \rangle = 0.587 \left(\frac{\mathcal{D}t}{\tau^2} \right)^{1/3}. \quad (61)$$

For $n=3$,

$$\langle E \rangle = 0.474 \left(\frac{\mathcal{D}t}{\tau^2} \right)^{3/5}, \quad \langle \dot{x}^2 \rangle = 0.711 \left(\frac{\mathcal{D}t}{\tau^2} \right)^{3/5}, \quad \langle x^2 \rangle = 0.535 \left(\frac{\mathcal{D}t}{\tau^2} \right)^{1/5}. \quad (62)$$

For $n=4$,

$$\langle E \rangle = 0.436 \left(\frac{\mathcal{D}t}{\tau^2} \right)^{4/7}, \quad \langle \dot{x}^2 \rangle = 0.698 \left(\frac{\mathcal{D}t}{\tau^2} \right)^{4/7}, \quad \langle x^2 \rangle = 0.500 \left(\frac{\mathcal{D}t}{\tau^2} \right)^{1/7}. \quad (63)$$

From Eq. (55), the skewness and the flatness of the energy PDF can also be calculated

$$S(E) = \frac{\langle E^3 \rangle}{\langle E^2 \rangle^{3/2}} = \frac{\Gamma\left(\frac{7n+1}{4n-2}\right) \Gamma\left(\frac{n+1}{4n-2}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{5n+1}{4n-2}\right)^{\frac{3}{2}}}, \quad (64)$$

$$F(E) = \frac{\langle E^4 \rangle}{\langle E^2 \rangle^2} = \frac{\Gamma\left(\frac{9n+1}{4n-2}\right) \Gamma\left(\frac{n+1}{4n-2}\right)}{\Gamma\left(\frac{5n+1}{4n-2}\right)^2}. \quad (65)$$

The skewness is approximatively equal to 1.95 for $n=2, 3$ and 4 and its limiting value for $n \rightarrow \infty$ is given by $\frac{\Gamma(\frac{7}{4})\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})^{3/2}} = \frac{6\sqrt{2}\pi}{\Gamma(\frac{1}{4})^2} \simeq 2.03 \dots$ The flatness

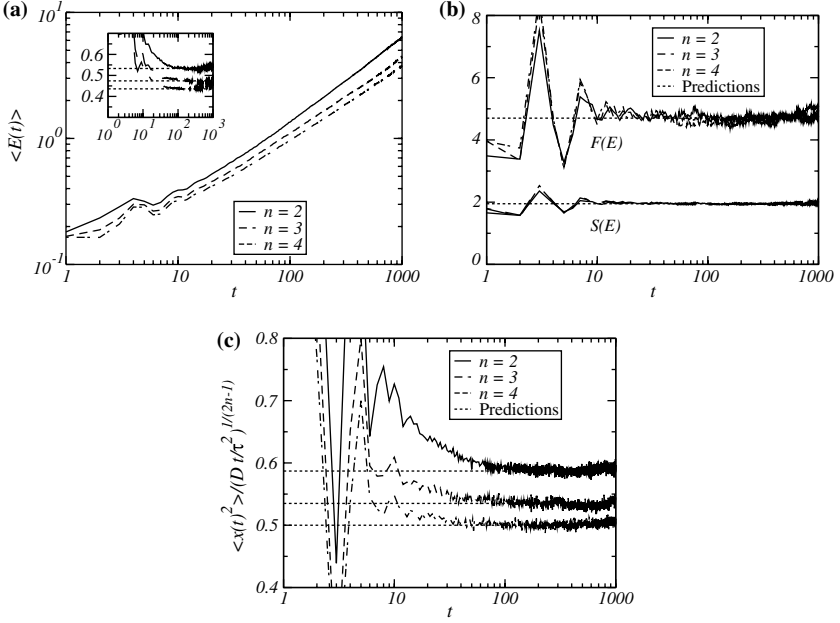


Fig. 3. Asymptotic behavior of the nonlinear oscillator with additive Ornstein-Uhlenbeck noise. Eqs. (3) and (20) are integrated numerically for $\mathcal{D}=1$, $\tau=5$, with a timestep $\delta t=10^{-5}$ and averaged over 10^4 realizations. For $n=2, 3, 4$, we plot: (a) the average $\langle E \rangle$ and the ratio $\langle E \rangle / (\mathcal{D}t/\tau^2)^{n/(2n-1)}$ (inset); (b) the skewness and flatness factors of E versus time t (c) the ratio $\langle x^2 \rangle / (\mathcal{D}t/\tau^2)^{1/(2n-1)}$. The dotted lines in the figures correspond to the analytical predictions, Eqs. (58), (60), (64) and (65).

is approximately equal to 4.7 for $n=2, 3$ and 4 and its limiting value for $n \rightarrow \infty$ is equal to 5.

The analytical results of Eqs. (61)–(65) are compared with numerical simulations in Fig. 3. We observe that the agreement is remarkable in the long time limit.

4.2. Extension to the Case of a Weakly Dissipative System

We now introduce a linear friction term in the system with dissipation rate γ . Equation (55) then becomes

$$\frac{d^2}{dt^2}x(t) + \gamma \frac{d}{dt}x(t) + x(t)^{2n-1} = \xi(t). \quad (66)$$

In terms of the (Ω, ϕ) variables defined in Eqs. (6), (7) and (9), Eq. (66) becomes

$$\dot{\Omega} = -\frac{n\gamma\mathcal{S}_n'^2(\phi)}{(2n)^{\frac{1}{n}}}\Omega + n\mathcal{S}_n'(\phi)\xi(t), \quad (67)$$

$$\dot{\phi} = \frac{\gamma\mathcal{S}_n(\phi)\mathcal{S}_n'(\phi)}{(2n)^{\frac{1}{n}}} + \left(\frac{\Omega}{(2n)^{\frac{1}{2n}}}\right)^{\frac{n-1}{n}} - \frac{\mathcal{S}_n(\phi)}{\Omega}\xi(t). \quad (68)$$

We want to average out the angular variable from this system; this is possible only if the evolution of ϕ is fast as compared to that of the energy: this condition is satisfied when the dissipation rate is vanishingly small so that it has a negligible effect during a period of the underlying deterministic oscillator. We proceed as in Section 3 and obtain after adiabatic elimination of the angle the following white noise Langevin equation for the effective variable Z_1 [defined in Eq. (51)]

$$\dot{Z}_1 = -\frac{(2n-1)\gamma}{n+1}Z_1 + (2n-1)(2-n)\frac{\mu\mathcal{D}}{2\tau^2}\frac{1}{Z_1} + (2n-1)\frac{\sqrt{\mu}}{\tau}\tilde{\eta}(t). \quad (69)$$

To derive this equation we have assumed that $\gamma\tau \ll 1$, and have used the identity $\overline{\mathcal{S}_n'^2} = \frac{(2n)^{\frac{1}{n}}}{n+1}$ (derived in Appendix A). When $t \rightarrow \infty$, the PDF of Z_1 tends to a well defined stationary limit $P_{\text{stat}}(Z_1)$ which can be explicitly calculated by solving the stationary Fokker–Planck equation associated with Eq. (69). Reverting to the energy variable E , we obtain the stationary distribution function of E

$$P_{\text{stat}}(E) = \frac{\frac{2n-1}{n}}{\Gamma\left(\frac{n+1}{4n-2}\right)} \left(\frac{4n^2\gamma\tau^2}{(2n-1)(n+1)\mu\mathcal{D}}\right)^{\frac{n+1}{4n-2}} E^{\frac{1-n}{2n}} \exp\left(-\frac{4n^2\gamma\tau^2 E^{\frac{2n-1}{n}}}{(2n-1)(n+1)\mu\mathcal{D}}\right). \quad (70)$$

We have thus derived, in the limit of vanishingly small dissipation, an analytical expression for the stationary probability distribution function of the energy of a nonlinear oscillator in presence of an additive Ornstein–Uhlenbeck noise. In contrast to the case of white noise, this stationary PDF is not the canonical Gibbs–Boltzmann distribution.

The statistical behavior of the position, velocity and energy of the system in the long time limit can now be derived by using the PDF (70) and Eqs. (6) and (7):

$$\langle E \rangle = \frac{\Gamma\left(\frac{3n+1}{4n-2}\right)}{\Gamma\left(\frac{n+1}{4n-2}\right)} \left(\frac{(2n-1)(n+1)\mu\mathcal{D}}{4n^2\gamma\tau^2}\right)^{\frac{n}{2n-1}}, \quad (71)$$

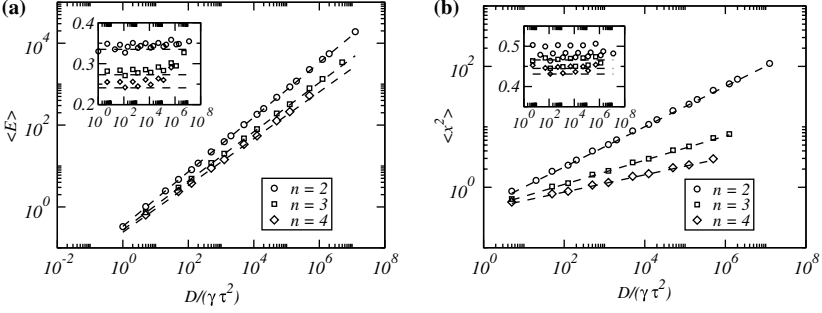


Fig. 4. The weakly dissipative case. Eq. (66) with Ornstein–Uhlenbeck noise is integrated numerically for $\gamma = 0.02$ with a timestep $\delta t = 10^{-5}$, and averaged over 10^3 realizations. For $n = 2, 3, 4$, we plot: (a) the average $\langle E \rangle$ (main plot) and the ratio $\langle E \rangle / (D/\gamma\tau^2)^{n/(2n-1)}$ (inset) versus $D/\gamma\tau^2$; (b) $\langle x^2 \rangle$ (main plot) and the ratio $\langle x^2 \rangle / (D/\gamma\tau^2)^{1/(2n-1)}$ (inset) vs. $D/\gamma\tau^2$. The parameters τ and D vary within the ranges $2 \leq \tau \leq 10$, $10^1 \leq D \leq 10^6$. The dashed lines in the figures correspond to the analytical predictions, Eqs. (71)–(73). Averages are evaluated beyond the dissipative and colored timescales ($t > 1/\gamma$ and $t > D^{-1} \tau^{-2n/(n-1)}$ respectively).

$$\langle \dot{x}^2 \rangle = \frac{2n}{n+1} \langle E \rangle, \quad (72)$$

$$\langle x^2 \rangle = \mu \langle E^{1/n} \rangle = \frac{\Gamma\left(\frac{n+3}{4n-2}\right)}{\Gamma\left(\frac{n+1}{4n-2}\right)} \left(\frac{(2n-1)(n+1)\mu^{2n}D}{4n^2\gamma\tau^2} \right)^{\frac{1}{2n-1}}. \quad (73)$$

We observe that Eqs. (70–73) can be obtained from Eqs. (57)–(60) by making the substitution $\frac{1}{2\tau} = \frac{2n-1}{n+1}\gamma$.

These analytical formulae compare favorably with numerical simulations (Fig. 4) for parameter values such that both γ and the product $\gamma\tau$ are sufficiently small (e.g., $\gamma = 0.02$, $2 \leq \tau \leq 10$). As expected, Eqs. (71)–(73) become less and less accurate as the dissipation coefficient γ becomes larger (however, the exponent $n/(2n-1)$ in (71) remains roughly valid at least up to $\gamma \simeq 1$). For smaller values of τ , care should be taken to ensure that measurements are indeed performed beyond the crossover time t_c where white-noise-like dynamical behavior is replaced by the time-asymptotic regime characteristic of colored noise ($t_c \sim D^{-1} \tau^{-2n/(n-1)}$) is a rapidly decreasing function of τ).

Lastly, our result for the stationary PDF in the weakly dissipative regime, (Eq. (70)), compares favorably with the PDF of a linear oscillator with Ornstein-Uhlenbeck noise⁽²²⁾

$$P_{\text{stat}}(\dot{x}, x) = \frac{\gamma(1+\gamma\tau+\tau^2)}{\pi D \sqrt{1+\gamma\tau}} \exp \left\{ -\frac{2\gamma}{D} \left(\frac{\dot{x}^2}{2} + \frac{x^2}{2} \frac{1}{1+\gamma\tau} \right) (1+\gamma\tau+\tau^2) \right\}. \quad (74)$$

In the limit $\tau \gg 1$ and $\gamma\tau \ll 1$, this expression reduces to

$$P_{\text{stat}}(E) = \frac{2\gamma\tau^2}{D} \exp\left(-\frac{2\gamma\tau^2 E}{D}\right), \quad (75)$$

which agrees with Eq. (70) for $n=1$.

5. CONCLUSION

We have studied the long time asymptotic behavior of a nonlinear oscillator subject to a Ornstein–Uhlenbeck noise. Because of the nonlinearity, the intrinsic time scale of the system becomes, in the long time limit, smaller than the correlation time of the noise. In this regime, the noise is intrinsically *colored* and the behavior of the system is radically different from that obtained for the white noise. The difference between white and colored noise driving can be quantitatively probed by calculating growth exponents that characterize the long time behavior of physical observables such as the energy, mean-square amplitude or velocity of the oscillator. We emphasize that the colored noise scalings are always satisfied in the long time limit however small be the correlation time τ of the noise. However, when τ is small, a crossover from the white noise scaling regime at short times to colored noise scalings at long times is observed. The crossover time t_c diverges to infinity as $\tau \rightarrow 0$. In other words, the limits $\tau \rightarrow 0$ and $t \rightarrow \infty$ (t being the observation time) do not commute.

In this work, our analysis is based on the exact Fokker–Planck equation obtained by embedding the initial problem into a higher dimensional Markovian system. In the long time limit, a dynamical separation of time-scales occurs: the angular variable becomes a fast variable whereas the energy evolves on a slower time scale. It is therefore tempting to eliminate the angle and derive an effective stochastic dynamics for the slow variable. Straightforward averaging leads to excellent results for the white noise case but it fails for colored noise. Indeed, at the lowest order the noise is practically constant during an angular period: thus, when the angular degree of freedom is integrated out, the noise itself is eliminated. A more sophisticated averaging procedure is therefore required that embodies the subdominant variations of the noise during the short time scale. By making recursive coordinate transformations on this higher dimensional system, we have managed to integrate out the fast variable while retaining the effect of the noise. We have finally derived an effective colored Langevin dynamics for the slow energy variable. At leading order, this effective dynamics is solved leading to an explicit formula for the probability distribution function of the energy in the long time limit. From this

PDF we have derived the moments of the position and the velocity of the oscillator. Our analytical results are compared with numerical simulations and the agreement is excellent. We may thus conjecture that the analytical expression (57) for the PDF of the energy, derived by our averaging scheme, is an exact asymptotic result.

We have extended our analysis to the case where the system is weakly dissipative; again the agreement between analytical and numerical results is very good. When the dissipation rate becomes higher, the agreement becomes less and less accurate; indeed, the angular variable can no more be considered to be fast as compared to the action variable and therefore it is not possible to use an adiabatic elimination procedure. However, in the regime of very large dissipation rate, the inertial term can be adiabatically eliminated^(23–25) and the problem is reduced to a first order Langevin equation with colored noise; many specific approximation schemes have been developed to study this case.^(26,27) The mathematical study of the intermediate dissipation regime remains to be done. Moreover, the extension of these averaging techniques to systems with spatio-temporal noise^(28,29) is a challenging open problem.

APPENDIX A: SOME USEFUL MATHEMATICAL IDENTITIES

In this Appendix, we derive some mathematical relations about the hyperelliptic function $\mathcal{S}_n(\phi)$ that have been used in the text. In Eq. (8), we defined the hyperelliptic function \mathcal{S}_n as

$$\mathcal{S}_n(\phi) = Y \leftrightarrow \phi = \sqrt{n} \int_0^{\frac{Y}{(2n)^{1/2n}}} \frac{du}{\sqrt{1-u^{2n}}} = \frac{\sqrt{n}}{(2n)^{1/2n}} \int_0^Y \frac{du}{\sqrt{1-\frac{u^{2n}}{2n}}}. \quad (\text{A1})$$

From this equation, we derive the following relation between \mathcal{S}_n and its derivative \mathcal{S}'_n

$$\mathcal{S}'_n(\phi) = \frac{(2n)^{\frac{1}{2n}}}{\sqrt{n}} \left(1 - \frac{(\mathcal{S}_n(\phi))^{2n}}{2n} \right)^{\frac{1}{2}}. \quad (\text{A2})$$

These hyperelliptic functions reduce to circular and elliptic functions for $n=1$ and $n=2$, respectively:

$$\mathcal{S}_1(\phi) = \sqrt{2} \sin \phi \quad \text{and} \quad \mathcal{S}'_1(\phi) = \sqrt{2} \cos \phi, \quad (\text{A3})$$

$$\mathcal{S}_2(\phi) = \frac{\operatorname{sn}\left(\phi; \frac{1}{\sqrt{2}}\right)}{\operatorname{dn}\left(\phi; \frac{1}{\sqrt{2}}\right)} \quad \text{and} \quad \mathcal{S}'_2(\phi) = \frac{\operatorname{cn}\left(\phi; \frac{1}{\sqrt{2}}\right)}{\operatorname{dn}^2\left(\phi; \frac{1}{\sqrt{2}}\right)}, \quad (\text{A4})$$

where sn , cn and dn are the classical Jacobi elliptic functions of modulus $1/\sqrt{2}$.^(14,15)

The functions \mathcal{S}_n and \mathcal{S}'_n are periodic functions of period K_n , defined in Eq. (5). Their average over a period vanishes. We also need to calculate the averages of the square of these functions. Using Eqs. (A1) and (5), we derive Eq. (48)

$$\mu = \overline{\mathcal{S}_n^2} = \frac{1}{K_n} \int_0^{K_n} \mathcal{S}_n^2(\phi) d\phi = (2n)^{\frac{1}{n}} \frac{\int_0^1 \frac{u^2 du}{\sqrt{1-u^{2n}}}}{\int_0^1 \frac{du}{\sqrt{1-u^{2n}}}} = (2n)^{\frac{1}{n}} \frac{\Gamma\left(\frac{3}{2n}\right) \Gamma\left(\frac{n+1}{2n}\right)}{\Gamma\left(\frac{1}{2n}\right) \Gamma\left(\frac{n+3}{2n}\right)}. \quad (\text{A5})$$

To obtain this identity, we made the change of variables $u = \mathcal{S}_n(\phi)$ and then $r = u^{2n}$, and finally used the Eulerian integral of the first kind⁽¹⁴⁾

$$\int_0^1 r^\alpha (1-r)^\beta dr = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}. \quad (\text{A6})$$

Similarly, from Eqs. (A2) and (5), and again making the change of variable $u = \mathcal{S}_n(\phi)$, we deduce that

$$\overline{\mathcal{S}'_n{}^2} = \frac{(2n)^{\frac{1}{n}}}{n} \frac{\int_0^1 du \sqrt{1-u^{2n}}}{\int_0^1 \frac{du}{\sqrt{1-u^{2n}}}}. \quad (\text{A7})$$

Using the following identity (that can be proved by integrating $\int_0^1 1 \cdot \sqrt{1-u^{2n}} du$ by parts)

$$\int_0^1 du \sqrt{1-u^{2n}} = n \int_0^1 du \frac{u^{2n}}{\sqrt{1-u^{2n}}} = -n \int_0^1 du \sqrt{1-u^{2n}} + n \int_0^1 \frac{du}{\sqrt{1-u^{2n}}}, \quad (\text{A8})$$

we conclude from Eq. (A7) that

$$\overline{\mathcal{S}'_n{}^2} = \frac{(2n)^{\frac{1}{n}}}{n+1}. \quad (\text{A9})$$

APPENDIX B: TRANSFORMATION OF THE ORIGINAL LANGEVIN EQUATIONS

In this Appendix, we perform a change of variables in the original Langevin equations and derive the stochastic dynamical system (39), (40) and (41) starting from the original Eqs. (24), (25) and (26). In terms of the variable Y defined in Eq. (33), Eq. (24) can be written as

$$\frac{\dot{Y}}{2n-1} = \frac{\dot{\Omega}}{n} \left(\frac{\Omega}{(2n)^{\frac{1}{2n}}} \right)^{\frac{n-1}{n}} = \left(\frac{Y}{(2n)^{\frac{1}{2n}}} \right)^{\frac{n-1}{2n-1}} S'_n(\phi) \xi. \quad (\text{B1})$$

Similarly, substituting the variable Y in Eq. (25) we obtain

$$\dot{\phi} = \left(\frac{Y}{(2n)^{\frac{1}{2n}}} \right)^{\frac{n-1}{2n-1}} \left(1 - \frac{S_n(\phi)\xi}{Y} \right). \quad (\text{B2})$$

From Eqs. (B1) and (B2), we deduce

$$\frac{\dot{Y}}{2n-1} = \frac{S'_n(\phi)\dot{\phi}\xi}{1 - \frac{S_n(\phi)\xi}{Y}}. \quad (\text{B3})$$

At long times, we know that $Y \rightarrow \infty$; thus we make a perturbative expansion of Eq. (B3) as follows

$$\frac{\dot{Y}}{2n-1} = S'_n(\phi)\dot{\phi}\xi + \frac{S'_n(\phi)S_n(\phi)\dot{\phi}\xi^2}{Y} + \frac{S'_n(\phi)S_n^2(\phi)\dot{\phi}\xi^3}{Y^2} + \dots \quad (\text{B4})$$

In the following calculations we retain terms up to the order $\mathcal{O}(Y^{-1})$ only, because $Y \rightarrow \infty$ in mean. In principle, random fluctuations may allow for small values of Y at arbitrary large times. However, in our numerical calculations, we never observed such fluctuations (of course, numerical simulations are carried out over a finite duration of time and can not be used as a proof that $Y \rightarrow \infty$ pathwise). Thus, the effective dynamics that we shall derive will be valid for large values of Y and the PDF that we shall obtain may not describe correctly the distribution of small values of Y . Nevertheless, since the relative probability of observing small values of Y tends to 0 as time grows, the formulae for the long time behavior of the moments will be asymptotically exact when $t \rightarrow \infty$.

Because we retain only contributions up to the order $\mathcal{O}(Y^{-1})$, we can neglect the last term of Eq. (B4): indeed, from Eq. (B2) we observe that

this term is of order $\mathcal{O}\left(\frac{\dot{\phi}}{Y^2}\right) = \mathcal{O}\left(Y^{\frac{1-3n}{2n-1}}\right)$, which is less than $\mathcal{O}(Y^{-1})$. We now integrate by parts the two relevant terms on the right hand side of Eq. (B4) in order to make the white noise contributions appear explicitly:

$$\begin{aligned} \frac{\dot{Y}}{2n-1} &= \frac{d}{dt} \left(\mathcal{S}_n(\phi) \xi + \frac{\mathcal{S}_n^2(\phi) \xi^2}{2Y} \right) + \frac{\mathcal{S}_n(\phi)(\xi + \eta)}{\tau} + \frac{\mathcal{S}_n^2(\phi) \xi(\xi + \eta)}{\tau Y} \\ &\quad + \frac{\mathcal{S}_n^2(\phi) \xi}{2} \frac{\dot{Y}}{Y^2} + \dots \end{aligned} \quad (\text{B5})$$

The last term in this equation is of the order $\mathcal{O}\left(\frac{\dot{Y}}{Y^2}\right) = \mathcal{O}\left(\frac{\dot{\phi}}{Y^2}\right)$ and is negligible with respect to $\mathcal{O}(Y^{-1})$. We thus obtain up to the desired order

$$\frac{d}{dt} \left\{ \frac{Y}{2n-1} - \mathcal{S}_n(\phi) \xi - \frac{\mathcal{S}_n^2(\phi) \xi^2}{2Y} \right\} = \frac{\mathcal{S}_n(\phi)(\xi + \eta)}{\tau} + \frac{\mathcal{S}_n^2(\phi) \xi(\xi + \eta)}{\tau Y}. \quad (\text{B6})$$

On the right hand side (r.h.s) of this equation, the white noise terms (proportional to η) and the term $\mathcal{S}_n^2(\phi) \xi^2 / \tau Y$ will survive the averaging over the angular variable ϕ whereas the term $\mathcal{S}_n(\phi) \xi / \tau$ will be eliminated because $\overline{\mathcal{S}_n(\phi)} = 0$. However, this term contains important correlations between ξ and ϕ of the order $\mathcal{O}(Y^{-1})$ and in order to retain these correlations, we transform this term as follows

$$\begin{aligned} \frac{\mathcal{S}_n(\phi) \xi}{\tau} &= \frac{\mathcal{S}_n(\phi) \xi}{\tau} \frac{\dot{\phi}}{\dot{\phi}} = \frac{\mathcal{S}_n(\phi) \dot{\phi} \xi}{\tau \left(\frac{Y}{(2n)^{\frac{1}{2n}}} \right)^{\frac{n-1}{2n-1}} \left(1 - \frac{\mathcal{S}_n(\phi) \xi}{Y} \right)} \\ &= \frac{(2n)^{\frac{n-1}{2n(2n-1)}} \mathcal{S}_n(\phi) \dot{\phi} \xi}{\tau Y^{\frac{n-1}{2n-1}}} + \frac{(2n)^{\frac{n-1}{2n(2n-1)}} \mathcal{S}_n^2(\phi) \dot{\phi} \xi^2}{\tau Y^{\frac{3n-2}{2n-1}}} + \dots \end{aligned} \quad (\text{B7})$$

Integrating by parts the first term on the r.h.s. of Eq. (B7) and using the function $C_n(\phi)$ defined in Eq. (38), we obtain

$$\begin{aligned} \frac{(2n)^{\frac{n-1}{2n(2n-1)}} \mathcal{S}_n(\phi) \dot{\phi} \xi}{\tau Y^{\frac{n-1}{2n-1}}} &= \frac{d}{dt} \left(\frac{(2n)^{\frac{n-1}{2n(2n-1)}} C_n(\phi) \xi}{\tau Y^{\frac{n-1}{2n-1}}} \right) + \frac{(2n)^{\frac{n-1}{2n(2n-1)}} C_n(\phi) (\xi + \eta)}{\tau^2 Y^{\frac{n-1}{2n-1}}} \\ &\quad + \frac{(2n)^{\frac{n-1}{2n(2n-1)}} (n-1) C_n(\phi) \xi \dot{Y}}{(2n-1) \tau Y^{\frac{3n-2}{2n-1}}} \end{aligned}$$

$$\begin{aligned}
&= \frac{d}{dt} \left(\frac{(2n)^{\frac{n-1}{2n(2n-1)}} C_n(\phi) \xi}{\tau Y^{\frac{n-1}{2n-1}}} \right) + \frac{(2n)^{\frac{n-1}{2n(2n-1)}} C_n(\phi) (\xi + \eta)}{\tau^2 Y^{\frac{n-1}{2n-1}}} \\
&\quad + \frac{(n-1) S'_n(\phi) C_n(\phi) \xi^2}{\tau Y}, \tag{B8}
\end{aligned}$$

where we have used Eq. (B1) to derive the last equality. Similarly, the second term on the r.h.s. of Eq. (B7) is rewritten as

$$\begin{aligned}
\frac{(2n)^{\frac{n-1}{2n(2n-1)}} S_n^2(\phi) \dot{\phi} \xi^2}{\tau Y^{\frac{3n-2}{2n-1}}} &= \frac{S_n^2(\phi) \xi^2}{\tau Y} \frac{\dot{\phi}}{\left(\frac{Y}{(2n)^{\frac{1}{2n}}} \right)^{\frac{n-1}{2n-1}}} = \frac{S_n^2(\phi) \xi^2}{\tau Y} + \mathcal{O}(Y^{-2}), \tag{B9}
\end{aligned}$$

where we have used Eq. (B2). If we substitute Eqs. (B8) and (B9) in Eq. (B7) and then substitute the result in Eq. (B6), we derive the following equation for Y , valid up to the order $\mathcal{O}(Y^{-1})$

$$\begin{aligned}
&\frac{d}{dt} \left\{ \frac{Y}{2n-1} - S_n(\phi) \xi - \frac{(2n)^{\frac{n-1}{2n(2n-1)}} C_n(\phi) \xi}{\tau Y^{\frac{n-1}{2n-1}}} - \frac{S_n^2(\phi) \xi^2}{2Y} \right\} \\
&= \frac{(2n)^{\frac{n-1}{2n(2n-1)}} C_n(\phi) \xi}{\tau^2 Y^{\frac{n-1}{2n-1}}} + \frac{2S_n^2(\phi) + (n-1) S'_n(\phi) C_n(\phi)}{\tau Y} \xi^2 \\
&\quad + \frac{1}{\tau} \eta(t) \left\{ S_n(\phi) + \frac{(2n)^{\frac{n-1}{2n(2n-1)}} C_n(\phi) \xi}{\tau Y^{\frac{n-1}{2n-1}}} + \frac{S_n^2(\phi) \xi}{Y} \right\}. \tag{B10}
\end{aligned}$$

The left hand side of Eq. (B10) is identical to $\frac{\dot{Z}}{2n-1}$, with Z defined in Eq. (37). On the right hand side of Eq. (B10), we must express Y as a function of Z . Although the relation (37) between Z and Y can not be inverted by a closed formula, Y can be calculated as a function of Z up to the order $\mathcal{O}(Z^{-1})$ included. This leads to Eq. (39) and to the formulae (42) and (43) for the current and diffusion functions.

APPENDIX C: DERIVATION OF THE AVERAGED FOKKER-PLANCK EQUATION

In this Appendix, we derive the averaged Fokker-Planck equation (47) by eliminating the fast angular variable from Eq. (45). We assume that the

PDF $\Pi_t(Z, \phi, \xi)$ becomes independent of ϕ when $t \rightarrow \infty$, i.e., the probability measure becomes uniform with respect to the angular variable in the long time limit. We integrate Eq. (45) with respect to the fast angular variable ϕ and examine the behavior of each term of Eq. (45). The average of the current $J_Z(Z, \phi, \xi)$ is given by:

$$\overline{J_Z} = \frac{(2n)^{\frac{n-1}{2n(2n-1)}} \overline{C_n(\phi)} \xi}{\tau^2 Z^{\frac{n-1}{2n-1}}} + \frac{2\overline{S_n^2(\phi)} + (n-1)\overline{S'_n(\phi)C_n(\phi)}}{\tau Z} \xi^2. \quad (\text{C1})$$

Integrating by parts and using Eq. (38), we obtain

$$\overline{S'_n(\phi)C_n(\phi)} = -\overline{S_n(\phi)C'_n(\phi)} = -\overline{S_n^2(\phi)}.$$

Using Eq. (48), we deduce that

$$\overline{J_Z} = \frac{3-n}{\tau Z} \mu \xi^2. \quad (\text{C2})$$

Therefore, we have

$$\frac{\partial}{\partial Z} \overline{(J_Z \Pi_t)} = \frac{\partial}{\partial Z} \left(\frac{(3-n)\xi^2}{\tau Z} \tilde{\Pi}_t \right). \quad (\text{C3})$$

Similarly, the average of the diffusion term $\mathcal{D}_Z(Z, \phi, \xi)$ is given by:

$$\overline{\mathcal{D}_Z} = \overline{S_n(\phi)} + \frac{(2n)^{\frac{n-1}{2n(2n-1)}} \overline{C_n(\phi)}}{\tau Z^{\frac{n-1}{2n-1}}} + \frac{\overline{S_n^2(\phi)} \xi}{Z} = \frac{\mu \xi}{Z}, \quad (\text{C4})$$

where we have used $\overline{S_n} = \overline{C_n} = 0$. The angular average of the expressions $\frac{\partial^2}{\partial Z \partial \xi} (\mathcal{D}_Z \Pi_t)$ and $\frac{\partial}{\partial Z} \mathcal{D}_Z \frac{\partial \Pi_t}{\partial \xi}$ that appear in Eq. (45) are readily deduced by using Eq. (C4):

$$\frac{\partial^2}{\partial Z \partial \xi} \overline{(\mathcal{D}_Z \Pi_t)} = \mu \frac{\partial^2}{\partial Z \partial \xi} \left(\frac{\xi \tilde{\Pi}_t}{Z} \right), \quad (\text{C5})$$

$$\frac{\partial}{\partial Z} \overline{\mathcal{D}_Z} \frac{\partial \tilde{\Pi}_t}{\partial \xi} = \mu \frac{\partial}{\partial Z} \left(\frac{\xi}{Z} \frac{\partial \tilde{\Pi}_t}{\partial \xi} \right). \quad (\text{C6})$$

Recalling that $\frac{\partial}{\partial Z}$ scales as Z^{-1} , the leading order in the expression $\frac{\partial}{\partial Z} D_Z \frac{\partial}{\partial Z} (D_Z \Pi_t)$ scales as Z^{-2} . Therefore, retaining only the leading term in the average of this expression, we obtain

$$\overline{\frac{\partial}{\partial Z} D_Z \frac{\partial}{\partial Z} (D_Z \Pi_t)} = \mu \frac{\partial^2 \tilde{\Pi}_t}{\partial Z^2}. \quad (C7)$$

Finally, using the fact that the integral of any term of the type $\partial_\phi(\dots)$ over a period of ϕ vanishes, we deduce

$$\overline{\frac{\partial}{\partial \phi} (J_\phi \Pi_t)} = 0. \quad (C8)$$

We have thus calculated the angular average of each term that appears on the right hand side of Eq. (45) [see Eqs. (C3), (C5), (C6), (C7) and (C8)]. This concludes the derivation of Eq. (47).

APPENDIX D: THE 'BEST FOKKER-PLANCK EQUATION' APPROXIMATION

In this appendix, we shall discuss an approximation called the BFPE which is based on a partial resummation of the perturbative small τ expansion.^(12,13) This BFPE has been rightly criticized⁽³⁰⁾ because it can lead to quantitatively incorrect results (e.g., in the study of the activation rate of overdamped bistable oscillators driven by colored noise). Nevertheless, we shall show that, for the system under study, the BFPE leads to the correct colored noise scaling behavior; it can also describe the crossover between white and colored noise regimes as we showed in ref. 11. Because the BFPE can not be expected to yield exact quantitative results, our discussion will be mostly qualitative.

Because random dynamical systems with colored noise are non-Markovian, there exists no closed Fokker-Planck equation that describes the dynamics of the PDF in phase-space. The Markovian character can be retrieved by embedding the system into a higher dimensional process, as has been done in the present work (see Section 3). Nevertheless, many approximation schemes have been developed to derive effective Fokker-Planck equations for systems subject to colored noise such as short correlation time expansions,⁽⁸⁻¹⁰⁾ functional methods,^(19,20) unified colored noise approximation (UCNA),⁽²²⁾ or decoupling Ansatz⁽²¹⁾ (for a review, see refs.16-18). We have compared some of these approximations with our averaging technique and our numerical results. We have observed that no short correlation time expansion truncated at any finite order could yield

the correct colored noise scalings of Eq. (23). Because, in the long time limit, the correlation time τ of the noise is not the shortest time scale of the system, any finite perturbative expansion with respect to τ is bound to fail. An expansion that embodies terms of all orders is needed: an approximate resummation of this type is provided by the BFPE.

The BFPE is constructed from an expansion of the stochastic Liouville equation associated with the system (10) and (11). These equations are rather involved due to the presence of hyperelliptic functions and because of the indirect coupling between Ω and ϕ through the noise term ξ . We can radically simplify these equations without altering their qualitative properties by writing

$$\dot{\Omega} = \kappa \cos \phi \xi(t), \quad (\text{D1})$$

$$\dot{\phi} = \left(\frac{\Omega}{(2n)^{\frac{1}{2n}}} \right)^{\frac{n-1}{n}}. \quad (\text{D2})$$

We have neglected the $1/\Omega$ term with respect to $\Omega^{(n-1)/n}$ in the equation for ϕ . Moreover, the hyperelliptic function $\mathcal{S}'_n(\phi)$ has been replaced by the ordinary circular function $\kappa \cos \phi$. The constant κ will be adjusted later so that the white noise limit of the system (D1) and (D2) coincides with that of the original system (10) and (11).

The second-order cumulant expansion⁽¹⁾ of the stochastic Liouville equation corresponding to Eqs. (10) and (11) is an approximate evolution equation for the PDF P_t in the small noise limit. It is given by

$$\frac{\partial P_t}{\partial t} = \mathbf{L}_0 P_t + \int_0^t dx \langle \mathbf{L}_1(t) \exp(\mathbf{L}_0 x) \mathbf{L}_1(t-x) \exp(-\mathbf{L}_0 x) \rangle P_t, \quad (\text{D3})$$

where the differential operators are defined as

$$\mathbf{L}_0 P_t = -\frac{\partial}{\partial \phi} \left(\left(\frac{\Omega}{(2n)^{\frac{1}{2n}}} \right)^{\frac{n-1}{n}} P_t \right), \quad (\text{D4})$$

$$\mathbf{L}_1(t) P_t = -\frac{\partial}{\partial \Omega} (\xi(t) \kappa \cos \phi P_t). \quad (\text{D5})$$

Following the procedure of⁽¹²⁾, we evaluate the right hand side of Eq. (D3) by applying the following operator formula

$$\exp(A) B \exp(-A) = B + [A, B] + \frac{1}{2!} [A, [A, B]] + \frac{1}{3!} [A, [A, [A, B]]] + \dots, \quad (\text{D6})$$

with $A = \mathbf{L}_0$ and $B = \mathbf{L}_1(t-x)$; we must therefore calculate iterated commutators of the operators \mathbf{L}_0 and \mathbf{L}_1 . By induction, it can be shown that the n -th commutator is of the form

$$\begin{aligned} T_k &= [\mathbf{L}_0, [\dots, [\mathbf{L}_0, [\mathbf{L}_0, \mathbf{L}_1(t-x)]] \dots]] \\ &= \kappa \xi(t-x) \left(\frac{\partial}{\partial \Omega} H_1^{(k)}(\Omega, \phi) + \frac{\partial}{\partial \phi} H_2^{(k)}(\Omega, \phi) \right), \end{aligned} \quad (\text{D7})$$

where the functions $H_1^{(k)}$ and $H_2^{(k)}$ satisfy the recursion relations

$$H_1^{(k)} = - \left(\frac{\Omega}{(2n)^{\frac{1}{2n}}} \right)^{\frac{n-1}{n}} \frac{\partial H_1^{(k-1)}}{\partial \phi}, \quad (\text{D8})$$

$$H_2^{(k)} = \frac{n-1}{n} \frac{\Omega^{-\frac{1}{n}}}{(2n)^{\frac{n-1}{2n^2}}} H_1^{(k-1)} - \left(\frac{\Omega}{(2n)^{\frac{1}{2n}}} \right)^{\frac{n-1}{n}} \frac{\partial H_2^{(k-1)}}{\partial \phi}, \quad (\text{D9})$$

and the initial conditions $H_1^{(0)} = -\cos \phi$, $H_2^{(0)} = 0$. The solution of these recursions is readily guessed once the first few terms have been calculated and we obtain

$$H_1^{(k)} = (-1)^{k-1} \left(\frac{\Omega}{(2n)^{\frac{1}{2n}}} \right)^{\frac{(n-1)k}{n}} \cos \left(\phi + k \frac{\pi}{2} \right), \quad (\text{D10})$$

$$H_2^{(k)} = (-1)^k \frac{(n-1)k}{n\Omega} \left(\frac{\Omega}{(2n)^{\frac{1}{2n}}} \right)^{\frac{(n-1)k}{n}} \sin \left(\phi + k \frac{\pi}{2} \right). \quad (\text{D11})$$

Using the expressions (D10) and (D11) for $H_1^{(k)}$ and $H_2^{(k)}$, respectively, and the autocorrelation function (21) of the Ornstein–Uhlenbeck noise, we calculate the integral on the right hand side of Eq. (D3)

$$\begin{aligned} &\int_0^t dx \langle \mathbf{L}_1(t) \exp(\mathbf{L}_0 x) \mathbf{L}_1(t-x) \exp(-\mathbf{L}_0 x) \rangle \\ &= -\frac{\mathcal{D}\kappa \cos \phi}{2\tau} \left(\frac{\partial^2}{\partial \Omega^2} \mathcal{H}_1(\Omega, \phi, t) + \frac{\partial^2}{\partial \Omega \partial \phi} \mathcal{H}_2(\Omega, \phi, t) \right), \end{aligned} \quad (\text{D12})$$

where we have defined

$$\begin{aligned}\mathcal{H}_1(\Omega, \phi, t) &= \sum_{k=0}^{\infty} \frac{\int_0^t dx x^k e^{-x/\tau}}{k!} H_1^{(k)} \quad \text{and} \\ \mathcal{H}_2(\Omega, \phi, t) &= \sum_{k=0}^{\infty} \frac{\int_0^t dx x^k e^{-x/\tau}}{k!} H_2^{(k)}\end{aligned}\quad (\text{D13})$$

In the limit $t \rightarrow \infty$, the integral $\int_0^{t/\tau} dx x^k e^{-x}$ converges to $k!$, and the series defining \mathcal{H}_1 and \mathcal{H}_2 can be calculated in a closed form:

$$\mathcal{H}_1(\Omega, \phi, \infty) = -\tau \frac{\cos \phi + \tau \omega \sin \phi}{1 + \tau^2 \omega^2}, \quad (\text{D14})$$

$$\mathcal{H}_2(\Omega, \phi, \infty) = -\tau^2 \frac{n-1}{n} \frac{\Omega^{-\frac{1}{n}}}{(2n)^{\frac{n-1}{2n^2}}} \frac{(1 - \tau^2 \omega^2) \cos \phi + 2\tau \omega \sin \phi}{(1 + \tau^2 \omega^2)^2}. \quad (\text{D15})$$

where we have defined

$$\omega = \left(\frac{\Omega}{(2n)^{\frac{1}{2n}}} \right)^{\frac{n-1}{n}}. \quad (\text{D16})$$

Substituting these expressions in Eq. (D12), we derive the BFPE associated with the system (D1) and (D2)

$$\begin{aligned}\frac{\partial P_t}{\partial t} &= -\frac{\partial}{\partial \phi} \left(\left(\frac{\Omega}{(2n)^{\frac{1}{2n}}} \right)^{\frac{n-1}{n}} P_t \right) + \frac{\mathcal{D}\kappa}{2} \frac{\partial^2}{\partial \Omega^2} \left(\frac{\cos^2 \phi + \tau \omega \sin \phi \cos \phi}{1 + \tau^2 \omega^2} P_t \right) \\ &\quad + \frac{\mathcal{D}\kappa \tau \cos \phi}{2} \frac{n-1}{n} \frac{\partial^2}{\partial \Omega \partial \phi} \left(\frac{\Omega^{-\frac{1}{n}}}{(2n)^{\frac{n-1}{2n^2}}} \frac{(1 - \tau^2 \omega^2) \cos \phi + 2\tau \omega \sin \phi}{(1 + \tau^2 \omega^2)^2} P_t \right)\end{aligned}\quad (\text{D17})$$

Integrating out the fast variable ϕ , we obtain an averaged BFPE for $\tilde{P}_t(\Omega)$, the probability distribution of the slow variable,

$$\frac{\partial \tilde{P}_t}{\partial t} = \frac{\mathcal{D}\kappa}{4} \frac{\partial}{\partial \Omega} \left(\frac{1}{1 + \tau^2 \omega^2} \frac{\partial \tilde{P}_t}{\partial \Omega} \right). \quad (\text{D18})$$

For $\tau=0$, this equation leads to a scaling behavior identical to the averaged white noise Fokker–Planck equation (15) and the average of Ω^2 agrees with Eq. (17) if we choose

$$\kappa = (2n)^{\frac{n+1}{n}}. \quad (\text{D19})$$

For a non-zero τ , the scaling behavior predicted by the averaged BFPE (D18) is indeed $\Omega \sim t^{n/(4n-2)}$ which is in agreement with Eq. (22). For $\tau \ll 1$, Eq. (D18) describes the crossover between the white noise regime at short times and the colored noise regime at long times⁽¹¹⁾.

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